DETERMINATION BY BOUNDING PRINCIPLES OF THE

COMBINED RADIATION FLUX ALONG THE SURFACE

BETWEEN TWO PARALLEL PLATES OF FINITE EXTENT

by

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NOMENCLATURE

dA_x	unit area on the surface of the plate
A_N	series coefficients in equations (13) and (14)
$B(x), B(y)$	combined radiant flux (emitted and reflected) leaving a position x or y per unit time and unit area
C_N	indefinite integral in equation (16)
C'	constant of integration in equation (16)
h	spacing between plates
$H(x)$	radiant energy arriving at x per unit time and area
H	gap spacing ratio, $\frac{h}{L/2}$
L	plate length
i, M, N	integers
$R_{N\delta}(X)$	residual function
T	absolute temperature
T_o	arbitrary reference temperature
$T_s(x)$	absolute temperature along the surface of the plate
$\mathcal{T}_s(X)$	surface temperature, $\frac{T_s(x)}{T_o}$
$\mathcal{T}'(X)$	temperature in equation (7)
$\mathcal{T}_l(X)$	lower bound temperature
$\mathcal{T}_u(X)$	upper bound temperature
U	$U = (Y-X)^2 + H^2$
U_{+1}	$U_{+1} = (1-X)^2 + H^2$
U_{-1}	$U_{-1} = (1+X)^2 + H^2$

x	co-ordinate measuring distance along lower plate from center plane
X	dimensionless co-ordinate, $\frac{x}{L/2}$
y	co-ordinate measuring distance along upper plate from center plane
Y	dimensionless co-ordinate, $\frac{y}{L/2}$
Z_N	definite integral in equation (15)
α	absorptivity
$\beta(X), \beta(Y)$	dimensionless combined flux, $\frac{B}{\epsilon\sigma T_o^4}$
$\beta^*(X), \beta^*(Y)$	radiant flux in equations (13) and (14)
$\beta_l(X), \beta_u(X), \beta_m(X)$	lower bound, upper bound, mean value
δ	parameter
δ_l, δ_u	parameter selected for lower bound, upper bound
ϵ	emissivity, $1 - \rho$
ρ	reflectivity, $1 - \epsilon$
σ	Stefan-Boltzmann's constant

INTRODUCTION

Today, more than ever before, the role of accurate error analysis in approximate procedures has become extremely vital. With the advent of high speed digital computers, the solution of many difficult and sometimes complex problems has been made possible using approximation techniques. The number of approximation techniques available are many. Unfortunately one does not have a definite idea of the error which is present using such techniques.

In this report, concerned with radiation heat transfer, the purpose is to formulate a procedure to determine bounds for the radiant flux along the surface between two parallel plates of finite extent. The results are compared with exact and other approximate methods. Once good upper and lower bounds have been obtained, the maximum possible error can be determined. Emphasis should be placed on the ability of this method to yield an approximate solution of known accuracy.

The bounding principle is a method in which convergent upper and lower bounds are obtained in order to approximate the solution to the problem. The formulation of the upper and lower bounds is established by bounding functions. These bounding functions essentially are intermediate problems to the actual problem. By varying these functions, the solution of the intermediate problem is always numerically greater than the exact solution in one case and always numerically lower than the exact solution in the other case. Applying this procedure systematically, the exact solution is then bounded above by a solution which is greater and bounded below by a solution which is less.

The method is practical in application since it is not necessary to obtain complete convergence. Once good bounds are established, it is not only possible to obtain a good approximate solution, but also to specify with certainty the maximum possible error. Thus, the difficult problem of error analysis is avoided.

The procedure for which the bounding principle is established is one which has been developed for determining bounds for the solution of differential equations. [1], [2], [3]

A radiation problem was chosen for the purpose of demonstrating the application of the bounding principles to integral equations which arise in radiation heat transfer problems. Since the solution of integral equations in general is a difficult task even for those who are familiar with them, a simple physical situation was selected for this problem. In this way, the essential features of the formulation are retained without the difficulties which would be involved in a complex geometrical arrangement.

[ ] Numbers in brackets designate References at end of report.

DESCRIPTION OF RADIANT FLUX PROBLEM

The problem is to determine the rate of the combined radiation flux along the surface between two finite, non-black, parallel plates.

Consider the figure below in which a sketch of the parallel plates is shown.

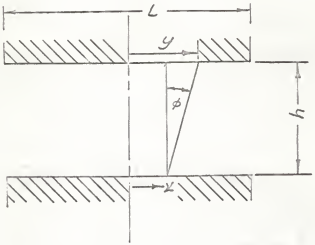

Fig. 1. Two Non-Black Parallel Plates.

Each plate is of length L. The plates are directly opposed and extend indefinitely in the direction normal to the plane of the figure. A distance h separates the two parallel plates. Each plate is considered as a gray-body radiator which both radiates and reflects in a diffuse manner. The gray-body emissivities of both plates are the same. Thus,

$$\epsilon = \alpha = 1 - \rho \tag{1}$$

The medium separating the two plates may contain either a non-participating gas or a vacuum.

The surface temperature of the plate is a function of the distance along the plate. For this problem the temperature is assumed constant and both plates are maintained at the same uniform temperature.

The net heat transfer from the system will be due to the radiant energy which is lost through the ends of the gap to the environment.

The rate of the combined radiation flux (emitted and reflected) leaving a unit area dA_x at x is designated as B(x). The emitted part of the flux is given by the well-known relation, $\epsilon\sigma T^4$. The amount of the reflected portion of the flux is $\rho H(x)$, where H(x) represents the radiant energy arriving at x per unit time and area. Thus the rate of the combined radiation flux can be expressed as

$$B(x) = \epsilon\sigma T_s^4(x) + \rho H(x) \tag{2}$$

(The B, H notation follows the notation used in illumination engineering and is the notation used by Eckert. [4])

Writing an energy balance for an elemental area of surface leads to the integral equation for the radiant flux between two non-black, parallel plates. [4] , [5] , [6] The integral equation is as follows

$$B(x) = \epsilon\sigma T_s^4(x) + \frac{\rho h^2}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} B(y)\frac{1}{[(y-x)^2+h^2]^{3/2}}dy \tag{3}$$

It may be noted here that the method used in most standard textbooks to determine the reflected portion of the radiant flux makes the assump-

tion that it is constant for all values of x rather than being a function of x.

The integral equation is transformed into a dimensionless form by dividing through by $\epsilon\sigma T_o^4$, where T_o is some arbitrarily chosen reference temperature. Thus, equation (3) becomes

$$\frac{B(x)}{\epsilon\sigma T_o^4} = \frac{\epsilon\sigma T_s^4(x)}{\epsilon\sigma T_o^4} + \frac{\rho h^2}{2}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{B(y)/\epsilon\sigma T_o^4}{\left[(y-x)^2+h^2\right]^{3/2}}\,dy \tag{3a}$$

Define the following expressions:

$$\beta(X) = \frac{B(x)}{\epsilon\sigma T_o^4} \tag{4}$$

$$\beta(Y) = \frac{B(y)}{\epsilon\sigma T_o^4} \tag{5}$$

$$\mathcal{J}_s(X) = \frac{T_s(x)}{T_o} \tag{6}$$

X and Y are dimensionless coordinates defined as $X = \frac{x}{L/2}$ and $Y = \frac{y}{L/2}$, respectively. Substituting these quantities into equation (3a) gives

$$\beta(X) = \mathcal{J}_s^4(X) + \frac{\rho h^2}{2}\int_{-1}^{1} \frac{\beta(Y)}{\left[\left(Y\cdot\frac{L}{2} - X\cdot\frac{L}{2}\right)^2 + h^2\right]^{3/2}}\,dY\cdot\frac{L}{2} \tag{3b}$$

The denominator of the integral expression becomes, after factoring an L/2 out of the expression

$$\left[\left(Y\cdot\frac{L}{2}-X\cdot\frac{L}{2}\right)^2+h^2\right]^{3/2} = \left[\left(\frac{L}{2}\right)^2\left(Y-X\right)^2+h^2\right]^{3/2}$$

which yields

$$\left(\frac{L}{2}\right)^3\left[\left(Y-X\right)^2+\left(\frac{h}{L/2}\right)^2\right]^{3/2}$$

Define H as the gap spacing ratio $H = \dfrac{h}{L/2}$ and substitute into equation (3b). This yields

$$\beta(X) = J_s^4(X) + \frac{\rho H^2}{2}\int_{-1}^{+1}\frac{\beta(Y)}{\left[(Y-X)^2+H^2\right]^{3/2}}\,dY \qquad (3c)$$

METHOD OF DETERMINING BOUNDS

Upper and lower bounds are established in terms of bounding functions. These bounding functions are essentially intermediate problems to the actual problem. The solution of the intermediate problem is always numerically greater than the exact solution in one case and always numerically lower than the exact solution in the other case. Systematic procedures can then be developed to provide for convergence of these bounding functions to the exact solution.

Let $\beta^*(X)$ represent a continuous function in the interval $0 \leq X \leq 1$. If $\beta^*(X)$ is substituted for $\beta(X)$ in the integral equation (3c), the equation will not be satisfied in general. Therefore, $\beta^*(X)$ is not, in general, the solution of the combined radiation flux problem with surface temperature, $\mathcal{J}_s(X)$. However, $\beta^*(X)$ does represent the exact solution of a problem corresponding to some other temperature. This will be denoted by $\mathcal{J}^*(X)$. Using equation (3c), $\mathcal{J}^*(X)$ can be determined explicitly by direct substitution of $\beta^*(X)$ for $\beta(X)$. Thus

$$\mathcal{J}^{*4}(X) = \beta^*(X) - \frac{\rho H^2}{2}\int_{-1}^{+1} \frac{\beta^*(Y)}{U^{3/2}}\, dY \tag{7}$$

where U is defined as $U = (Y - X)^2 + H^2$. It becomes apparent from equation (7) that for a given solution, the determination of the exact problem is a straight forward procedure.

Suppose that a particular $\beta^*(X) = \beta_u(X)$ can be chosen such that the corresponding temperature $\mathcal{J}^*(X) = \mathcal{J}_u(X)$ satisfies the following inequality:

$$\mathcal{J}_s(X) \leqq \mathcal{J}_u(X) \qquad 0 \leqq X \leqq 1 \qquad (8)$$

The net radiant flux distribution $\beta_u(X)$ corresponds to a situation where the surface temperature is greater than the desired temperature, $\mathcal{J}_s(X)$. Intuitively, one would conclude that the effect of an increase in the temperature would cause a corresponding increase in the net radiant flux. This can be expressed as

$$\beta(X) \leqq \beta_u(X) \qquad 0 \leqq X \leqq 1 \qquad (9)$$

Similarly if $\beta^*(X) = \beta_l(X)$ can be chosen such that $\mathcal{J}^*(X) = \mathcal{J}_l(X)$ satisfies

$$\mathcal{J}_l(X) \leqq \mathcal{J}_s(X) \qquad 0 \leqq X \leqq 1 \qquad (10)$$

then it follows that

$$\beta_l(X) \leqq \beta(X) \qquad 0 \leqq X \leqq 1 \qquad (11)$$

From equations (9) and (11), upper and lower bounds for $\beta(X)$ are thus established.

$$\beta_l(X) \leqq \beta(X) \leqq \beta_u(X) \qquad 0 \leqq X \leqq 1 \qquad (12)$$

The problem now becomes one of determining a practical procedure to obtain good upper and lower bounds in such a manner that the deviation is sufficiently small. The answer is to begin by choosing a $\beta^*(X) = \beta_u(X)$ such that two requirements are satisfied. One, $\mathcal{J}_u(X)$ should be approximately equal to $\mathcal{J}_s(X)$ and, two, $\mathcal{J}_s(X) \leq \mathcal{J}_u(X)$. Likewise, the choice $\beta^*(X) = \beta_l(X)$ must be made such that $\mathcal{J}_l(X) \approx \mathcal{J}_s(X)$ subject to the condition, $\mathcal{J}_l(X) \leq \mathcal{J}_s(X)$. Moreover, emphasis should be placed upon the fact that to obtain convergent bounds not only must $\mathcal{J}_u(X)$ and $\mathcal{J}_l(X)$ approach $\mathcal{J}_s(X)$, but the inequalities of equations (8) and (10) must be satisfied during the limiting process. It is apparent that the choice of $\beta^*(X)$ must result in a one-sided convergence for each bound.

The first step in determining bounds is to assume a series form of the solution for $\beta^*(X)$. Due to the symmetry of the problem, assume a power series solution for $\beta^*(X)$ and $\beta^*(Y)$ in equation (7) to be

$$\beta^*(X) = \sum_{N=0,2,4,\cdots}^{M} A_N X^N \tag{13}$$

and

$$\beta^*(Y) = \sum_{N=0,2,4,\cdots}^{M} A_N Y^N \tag{14}$$

Substitution into equation (7) leads to

$$\mathcal{J}^{*4}(X) = \sum_{N=0,2,4,\cdots}^{M} A_N X^N - \frac{\rho H^2}{2} \sum_{N=0,2,4,\cdots}^{M} A_N \int_{-1}^{+1} \frac{Y^N}{U^{3/2}}\, dY \tag{7a}$$

The definite integrals are now defined as

$$Z_N = \int_{-1}^{+1} \frac{Y^N}{U^{3/2}}\, dY \tag{15}$$

and the indefinite integrals are written as

$$C_N = \int \frac{Y^N}{U^{3/2}}\, dY + C' \tag{16}$$

From a table of integrals [7], an equation is written for each value of N in equation (16) for N = 0, 1, 2. A general equation is then developed for any value greater than 2. The evaluation of the indefinite integrals is given in Appendix A. The equations for C_N are as follows:

$$C_{N0} = \frac{Y - X}{H^2 U^{1/2}} \tag{16a}$$

$$C_{N1} = \frac{XY - X^2 - H^2}{H^2 U^{1/2}} \tag{16b}$$

$$C_{N2} = \ln(Y - X + U^{1/2}) + \frac{X^2Y - X^3 - XH^2 - YH^2}{H^2 U^{1/2}} \tag{16c}$$

$$C_{N>2} = \frac{Y^{N-1}}{(N-2)U^{1/2}} + \frac{(2N-3)X}{N-2} C_{N-1} + \frac{(1-N)(X^2+H^2)}{N-2} C_{N-2} \tag{16d}$$

Evaluating the definite integral in equation (15) for N = 0 at its limits of -1 and +1 gives

$$Z_{N0} = C_{N0}(X, 1.0) - C_{N0}(X, -1.0)$$

$$Z_{N0} = \frac{1 - X}{H^2 U_{+1}^{1/2}} + \frac{1 + X}{H^2 U_{-1}^{1/2}} \tag{17}$$

where $U_{+1} = (1 - X)^2 + H^2$ and $U_{-1} = (1 + X)^2 + H^2$. For N = 1, the equation is

$$Z_{N1} = C_{N1}(X, 1.0) - C_{N1}(X, -1.0)$$

$$Z_{N1} = \frac{X - X^2 - H^2}{H^2 U_{+1}^{1/2}} + \frac{X + X^2 + H^2}{H^2 U_{-1}^{1/2}} \tag{18}$$

For N = 2

$$Z_{N2} = C_{N2}(X, 1.0) - C_{N2}(X, -1.0)$$

$$Z_{N2} = \ln(1 - X + U_{+1}^{1/2}) - \ln(1 - X + U_{-1}^{1/2})$$

$$+ \frac{X^2 - X^3 - XH^2 - H^2}{H^2\, U_{+1}^{1/2}} + \frac{X^2 + X^3 + XH^2 - H^2}{H^2\, U_{-1}^{1/2}} \qquad (19)$$

Finally, $N > 2$ leads to

$$Z_{N>2} = C_{N>2}(X, 1.0) - C_{N>2}(X, -1.0)$$

$$Z_{N>2} = \frac{1}{(N-2)\, U_{+1}^{1/2}} - \frac{(-1)^{N-1}}{(N-2)\, U_{-1}^{1/2}} \qquad (20)$$

Examining equation (7a), the next step becomes one of determining the coefficients, A_i , from the assumed power series solution for $\beta^*(X)$. This is accomplished by using the method of collocation.

The method of collocation is a relatively simple and straight forward approximation method which in essence is a weighted residual procedure. It is used here to "force" $\mathcal{J}_u(X)$ and $\mathcal{J}_l(X)$ to be approximately equal to $\mathcal{J}_s(X)$ in the interval $0 \leq X \leq 1$. It is not satisfactory, in general, to let $\mathcal{J}_u(X)$ and $\mathcal{J}_l(X)$ approach $\mathcal{J}_s(X)$ in the usual manner. In order that upper and lower bounds may be established, the requirement that the inequalities of equations (8) and (10) must also be maintained. It then becomes convenient, therefore,

to let $\mathcal{J}_u(X)$ and $\mathcal{J}_l(X)$ approach a new value, $\mathcal{J}_s(X) + \delta$, where δ is a parameter. By suitable choices of δ, the inequalities can be maintained. Also, it is convenient to define a residual function, $R_{N\delta}(X)$, as

$$R_{N\delta}(X) = \mathcal{J}^{*4}(X) - \mathcal{J}_s^{4}(X) \tag{21}$$

The method of collocation consists of selecting a number of equally spaced points for X in the interval $0 \leq X \leq 1$ and equating the values of the residuals to δ at each one of these points. This leads to a system of simultaneous equations,

$$R_{N\delta}(X_i) - \delta = 0 \tag{22}$$

where $i = 1, 2, \ldots, (\frac{M}{2} + 1)$. The A_i's can then be determined from any matrix solution program.

Once the coefficients have been obtained, the next step in the bounding technique is to consider the residual equation (21) again. To insure bounds, the residuals must be checked to determine if the proper inequality has been satisfied at all points in the interval $0 \leq X \leq 1$. For each value of δ, the residuals are examined at 101 points to determine the approximate maximum and minimum values of the residual.

An upper bound for the net radiant flux is obtained by choosing a sufficiently large positive value of δ. The residuals and the A's are then calculated. If the minimum residual is positive, then the inequality (8) is satisfied and the A's can be substituted into equation (7). The resulting

$\beta_u(X)$ is an upper bound for $\beta(X)$. If the bound scheme is repeated for successively smaller values of δ and the minimum residual approaches zero but still remains positive, a least upper bound will be attained for $\beta(X)$.

The lower bound for the radiant flux is similarly obtained by choosing a sufficiently large negative value of δ and determining the residuals and the A's. If the maximum residual is negative, then the inequality (10) is maintained and the A's can be substituted into equation (7). The resulting $\beta_l(X)$ is a lower bound for $\beta(X)$. Better lower bounds can be obtained by letting δ approach zero and verifying that the sign of the maximum residual remains negative.

A schematic for illustrating how the maximum and minimum residuals vary with different values of δ is shown.

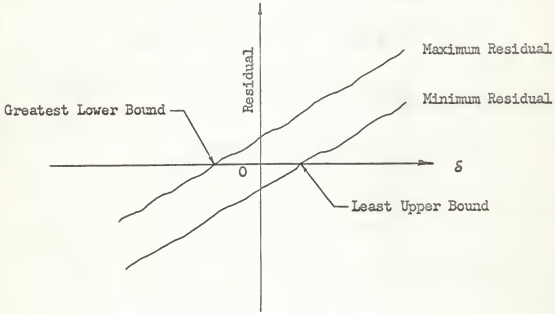

Fig. 2. Maximum and Minimum Residuals versus δ.

Based upon the definition of the residual, it can be seen that for the least upper bound and the greatest lower bound, the inequalities of equations (8) and (10) are maintained.

The best approximation to the exact solution is the mean value of the upper and lower bounding solutions. Thus

$$\beta_m(X) = \frac{\beta_u(X) + \beta_L(X)}{2} \tag{23}$$

is the mean value. The maximum percent error associated with this mean value is expressed as

$$\text{Maximum Percent Error} \qquad = \frac{\beta_m(X) - \beta_L(X)}{\beta_L(X)}(100) \tag{24}$$

NUMERICAL RESULTS

The numerical results obtained for the net radiant flux along the surface of the plate are compared with a solution obtained by direct iteration on an electronic computer. The iterative solution is given by Sparrow. [6] This solution is used as a standard. In all of the cases considered, $\mathcal{J}_s(X) = 1.0$ to agree with the work in [6]. The numerical results, corresponding to three cases, are given in the Tables I, II and III. Table IV shows the coefficients of the bounding functions for the three cases considered. The δ's selected in the three cases are the best values for obtaining good upper and lower bounds. Figure 3 shows a comparison of the mean value and the exact solution for a

particular case. Figure 4 shows how the bounds vary along the surface of the plate for different values of δ.

The numerical work was done on an IBM 1410 digital computer using 8 place arithmetic.

Five terms were selected in the series solution for $\beta^*(X)$ and $\beta^*(Y)$ in equations (13) and (14).

Table I

Comparison of Bounds with Exact* Solution

Collocation: $N = 5$; $H = 2.0$; $\rho = 0.9$; $\mathcal{J}_s(X) = 1.0$

X	Lower Bound ($\delta = -0.1 \times 10^{-4}$)	Upper Bound ($\delta = 0.1 \times 10^{-4}$)	Mean Value	Maximum Possible Error ($\% \times 10^4$)	Exact*
0.0	1.669693	1.669705	1.669699	3.623	1.644
0.1	1.668101	1.668113	1.668107	3.626	
0.2	1.663352	1.663364	1.663358	3.667	1.638
0.3	1.655523	1.655535	1.655529	3.714	
0.4	1.644744	1.644756	1.644750	3.708	1.620
0.5	1.631193	1.631205	1.631199	3.647	
0.6	1.615094	1.615106	1.615100	3.622	1.590
0.7	1.596712	1.596724	1.596718	3.663	
0.8	1.576347	1.576360	1.576354	3.837	1.552
0.9	1.554326	1.554339	1.554332	4.117	
1.0	1.530988	1.530999	1.530994	3.690	1.508

* Exact solution is based upon a direct iteration. [6]

Table II

Comparison of Bounds with Exact* Solution

Collocation: $N = 5$; $H = 1.0$; $\rho = 0.9$; $\mathcal{J}_s(X) = 1.0$

X	Lower Bound ($\delta = -0.3 \times 10^{-3}$)	Upper Bound ($\delta = 0.3 \times 10^{-3}$)	Mean Value	Maximum Possible Error ($\% \times 10^2$)	Exact*
0.0	2.507274	2.507776	2.507525	1.001	2.485
0.1	2.500794	2.501294	2.501044	1.001	
0.2	2.481375	2.481872	2.481623	1.001	2.459
0.3	2.449105	2.449596	2.449350	1.001	
0.4	2.404201	2.404682	2.404441	1.001	2.383
0.5	2.347100	2.347570	2.347335	1.001	
0.6	2.278584	2.279040	2.278812	1.001	2.259
0.7	2.199898	2.200338	2.200118	1.001	
0.8	2.112858	2.113281	2.113070	1.001	2.095
0.9	2.019922	2.020326	2.020124	1.001	
1.0	1.924179	1.924564	1.924371	1.000	1.908

* Exact solution is based upon a direct iteration. [6]

Table III

Comparison of Bounds with Exact* Solution

Collocation: $N = 5$; $H = 0.2$; $\rho = 0.9$; $\mathcal{J}_s(X) = 1.0$

X	Lower Bound ($\delta = -0.725 \times 10^{-1}$)	Upper Bound ($\delta = 0.2 \times 10^{-2}$)	Mean Value	Maximum Possible Error (%)	Exact*
0.0	7.040925	7.606476	7.323700	4.016	7.22
0.1	7.023116	7.587236	7.305176	4.016	
0.2	6.969220	7.529012	7.249116	4.016	7.11
0.3	6.877762	7.430207	7.153984	4.016	
0.4	6.745751	7.287592	7.016672	4.016	6.75
0.5	6.566988	7.094375	6.830637	4.016	
0.6	6.327145	6.835362	6.581253	4.016	6.07
0.7	5.995503	6.477080	6.236292	4.016	
0.8	5.507702	5.950098	5.728900	4.016	4.90
0.9	4.739498	5.120189	4.929844	4.016	
1.0	3.465979	3.744378	3.605178	4.016	2.97

* Exact solution is based upon a direct iteration. [6]

Table IV

Coefficients of the Bounding Functions

Collocation $N = 5$; $H = 2.0$; $\rho = 0.9$; $\mathcal{J}_s(X) = 1.0$

	Lower Bound ($\delta = -0.1 \times 10^{-4}$)	Upper Bound ($\delta = 0.1 \times 10^{-4}$)
A_1	0.16696929×10^{1}	0.16697050×10^{1}
A_2	-0.15940978	-0.15940411
A_3	$0.21894532 \times 10^{-1}$	$0.21851796 \times 10^{-1}$
A_4	$-0.97139590 \times 10^{-3}$	$-0.88792943 \times 10^{-3}$
A_5	$-0.21857891 \times 10^{-3}$	$-0.26551958 \times 10^{-3}$

Collocation $N = 5$; $H = 1.0$; $\rho = 0.9$; $\mathcal{J}_s(X) = 1.0$

	Lower Bound ($\delta = -0.3 \times 10^{-3}$)	Upper Bound ($\delta = 0.3 \times 10^{-3}$)
A_1	0.25072739×10^{1}	0.25077759×10^{1}
A_2	-0.64818792	-0.64831781
A_3	$0.15363745 \times 10^{-1}$	$0.15368850 \times 10^{-1}$
A_4	$0.61335306 \times 10^{-1}$	$0.61343168 \times 10^{-1}$
A_5	$-0.11606111 \times 10^{-1}$	$-0.11606003 \times 10^{-1}$

Table IV (continued)

Collocation $N = 5$; $H = 0.2$; $\rho = 0.9$; $\mathcal{J}_s(X) = 1.0$

	Lower Bound ($\delta = -0.725 \times 10^1$)	Upper Bound ($\delta = 0.2 \times 10^{-2}$)
A_1	0.70409250×10^1	0.76064761×10^1
A_2	-0.17770760×10^1	-0.19198206×10^1
A_3	-0.38609392	-0.41710001
A_4	$-0.94682988 \times 10^{-2}$	$-0.10241784 \times 10^{-1}$
A_5	-0.14023075×10^1	-0.15149361×10^1

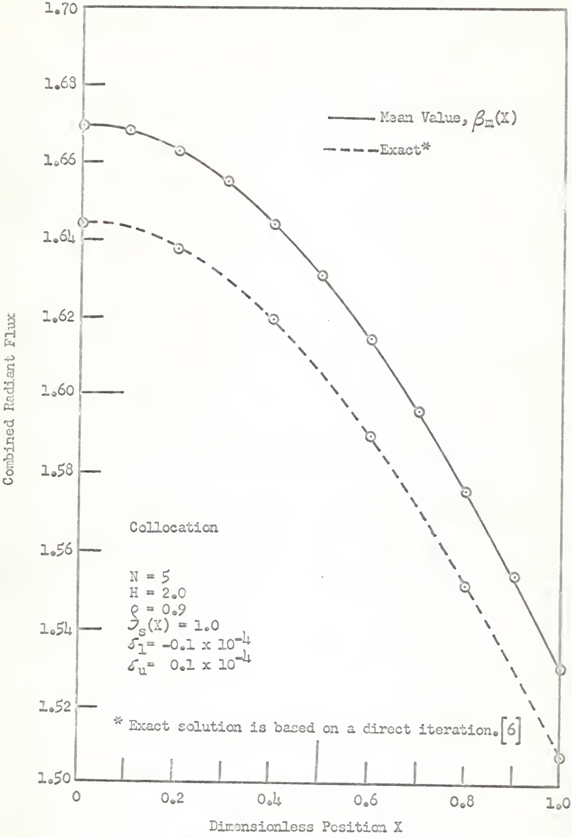

Fig. 3. Comparison of Mean Value and Exact* Solution.

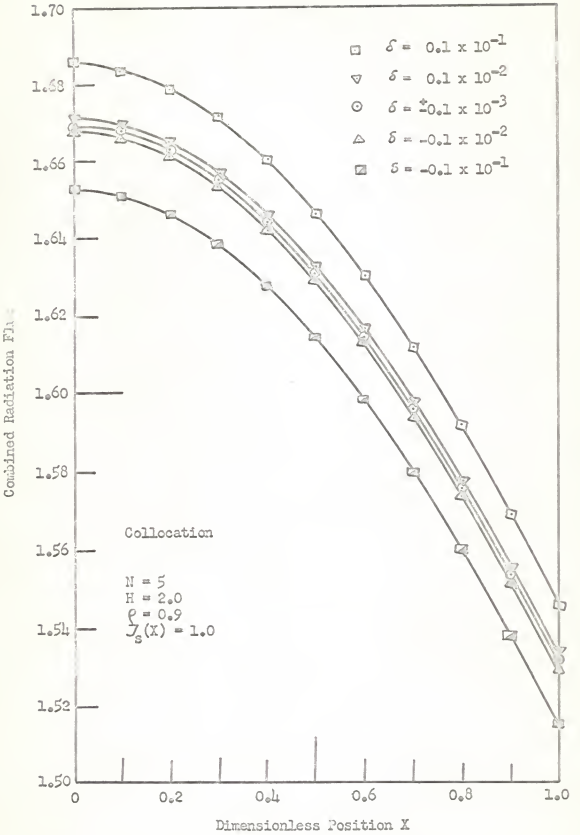

Fig. 4. Comparison of Bounds versus δ.

DISCUSSION OF RESULTS

The present results indicate that the convergence of the upper and lower bounds is obtained for the combined radiation flux along the surface of the plate. Bounds are established for each of the three cases considered. The percent error remains relatively constant along the surface of the plate for each case. This indicates that the deviation between the bounds is constant along the surface.

As the gap spacing ratio, H, is decreased from 2.0 to 0.2, the maximum percent error becomes larger. Thus, it is concluded that better bounds are obtained for larger gap spacing ratios.

A comparison of the results obtained and the exact solution shows that they are not in agreement. It can be seen in Fig. 3 that the shape of the curve for both solutions is approximately the same. The solutions appear to differ by a constant. This implies that the disagreement is not too serious and is possibly due to either a systematic or numerical error. Further investigation is recommended to determine the nature of the disagreement.

In the first case considered, it is shown in Fig. 4 that as the absolute value of δ becomes increasingly smaller, better bounds are obtained for the net radiant flux. The values of the radiant flux for $\delta = 0.1 \times 10^{-3}$ and $\delta = -0.1 \times 10^{-3}$ are so near being equal to each other that the curves are coincident.

It was interesting to note that plots of the maximum and minimum residuals versus δ are shown to be straight lines.

In looking for methods to reduce the error associated with the N-th variation in the assumed form of the power series solution, it is recommended to increase the value of N. The result will cause the bounds to

approach the exact solution with greater accuracy. In the formulation of this problem, note that the values of N are even integers.

The surface temperature of the plate, $\mathcal{J}_s(X)$, was assumed constant for the purpose of comparing the results with other available approximate and exact solutions. With the present program, the plate temperature may be varied at different positions along the plate.

CONCLUSIONS

Although complete agreement of the solutions with the exact solution has not been obtained, the results indicate that the bounding principle, applied to integral equations, gives convergent upper and lower bounds. The procedure can be extended to other problems which are described by integral equations. Integration of equation (16) is a relatively simple operation for the function Y and U. However, the solution becomes more complex as the geometry of the problem is varied.

It is believed that the application of the bounding principles will be a welcome addition to the field of radiation space technology and to the solution of integral equations encountered in radiant heat transfer problems. From the formulation presented in this paper, it is now possible to determine both the local and the overall heat transfer from the system.

When the plates which exchange heat are non-gray radiators, the equations must be written for monochromatic radiation.

The degree of accuracy of the solution obtained by bounding principles is dependent upon the efficiency and speed of present day high speed digital computing facilities. The method is practical and affords accurate results. Moreover, the use of this method eliminates the need for "error analysis".

ACKNOWLEDGEMENT

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APPENDIX A

Evaluation of the Indefinite Integral

EVALUATION OF THE INDEFINITE INTEGRAL

For the indefinite integral of equation (15), Gröbner [7] has given the integral in the form

$$\int \frac{Y^N}{U^{k+1/2}}\,dY = \frac{A\,Y^{N-1}}{U^{k-1/2}} + B\int \frac{Y^{N-1}}{U^{k+1/2}}\,dY + D\int \frac{Y^{N-2}}{U^{k+1/2}}\,dY \qquad (25)$$

where $N \neq 2k$ and $A = \dfrac{1}{(N - 2k)a}$; $B = -\dfrac{(2N - 2k - 1)b}{(N - 2k)a}$;

$$D = -\frac{(N - 1)c}{(N - 2k)a}$$

The denominator U is a quadratic equation expressed as

$$U = aY^2 + 2bY + c \qquad (26)$$

For the radiant flux problem between two parallel plates

$$U = (Y - X)^2 + H^2 \qquad (27)$$

and upon expanding

$$U = Y^2 - 2XY + (X^2 + H^2) \qquad (27a)$$

Thus the quantities a, b, and c are a = 1, b = $-X$, and c = $X^2 + H^2$. The value of k in equation (25) is k = 1 so that $U^{k+\frac{1}{2}} = U^{\frac{3}{2}}$.

Recall that the requirement $N \neq 2k$ must be satisfied for equation (25). Since k = 1, then it follows that $N \neq 2$. The effect of this requirement indicates that equation (25) can be used for any value of N in the power series expansion of the dummy variable Y in equation (7a) except at the value N = 2. This author elected to use the integral equation for values of N where $N > 2$ and to evaluate the integral at N = 0, 1 and 2 by a standard table of integrals. [7] For N = 0, the integral equation becomes

$$\int \frac{dY}{U^{3/2}} = \frac{1}{ac - b^2} \frac{aY + b}{U^{1/2}} = \frac{Y - X}{H^2 U^{1/2}} \tag{28}$$

For N = 1

$$\int \frac{Y\,dY}{U^{3/2}} = \frac{-1}{ac - b^2} \frac{bY + c}{U^{1/2}} = \frac{XY - X^2 - H^2}{H^2 U^{1/2}} \tag{29}$$

Finally, N = 2 results in

$$\int \frac{Y^2 dY}{U^{3/2}} = \frac{1}{a} \int \frac{dY}{U^{1/2}} + \frac{1}{a(ac - b^2)} \frac{(2b^2 - ac)Y + bc}{U^{1/2}} \tag{30}$$

$$\int \frac{dY}{U^{1/2}} = \frac{1}{\sqrt{a}} \ln \left(\frac{aY+b}{\sqrt{a}} + U^{1/2} \right) \tag{31}$$

Substituting equation (31) into equation (30) yields

$$\int \frac{Y^2\,dY}{U^{3/2}} = \ln \left(Y - X + U^{1/2} \right) + \frac{X^2 Y - X^3 - XH^2 - YH^2}{H^2\, U^{1/2}} \tag{30a}$$

APPENDIX B

Listing of Fortran Program to Solve for Coefficients, A_i.

```
KSU 1410 COMPUTING CENTER

               BOP  DIEGO RATH
C      COEFFICIENT MATRIX FOR RADIATION PROBLEM
   10 DIMENSION B(16,17)
    1 FORMAT(5E16.8)
    2 FORMAT(1H0,5E16.8)
    3 FORMAT(I3)
   15 READ INPUT TAPE 5,3,M
   16 READ INPUT TAPE 5,1,DELX
   17 READ INPUT TAPE 5,1,XQ
   18 READ INPUT TAPE 5,1,XT
   19 READ INPUT TAPE 5,1,H
   20 READ INPUT TAPE 5,1,RHO
   22 READ INPUT TAPE 5,1,ZAC,ZCORR
   23 READ INPUT TAPE 5,1,T
   24 READ INPUT TAPE 5,3,NIN
   25 READ INPUT TAPE 5,3,NDELTA
      GIN=NIN
      DELX1=1.0/GIN
      NIN1=NIN+1
      M2=(M/2)+1
      M3=M2+1
      LN=M2
      DO 60 MID=1,NDELTA
   21 READ INPUT TAPE 5,1,DELTA
    9 FORMAT(1H0,5X,20H DELTA IS NOW SET TO,E16.8)
      WRITE OUTPUT TAPE 6,9,DELTA
    5 FORMAT(1H0,7X,4HDELX,13X,2HXQ,15X,2HXT,14X,1HH,14X,3HRHO)
      WRITE OUTPUT TAPE 6,5
      WRITE OUTPUT TAPE 6,2,DELX,XQ,XT,H,RHO
    6 FORMAT(1H0,7X,5HDELTA,11X,3HZAC,13X,5HZCORR,13X,1HT)
      WRITE OUTPUT TAPE 6,6
      WRITE OUTPUT TAPE 6,2,DELTA,ZAC,ZCORR,T
    7 FORMAT(1H0,7X,1HM,5X,3HNIN,5X,6HNDELTA)
      WRITE OUTPUT TAPE 6,7
    8 FORMAT(1H0,5X,I3,5X,I3,6X,I3)
      WRITE OUTPUT TAPE 6,8,M,NIN,NDELTA
      X=XQ
      DM=M
      DELX=2.*(XT-XQ)/DM
      DO 50 I=1,M2
      B(I,M3)=DELTA+T**4
      N=2
      UPO=SQRTF((1.-X)*(1.-X)+H*H)
      UNE=SQRTF((1.+X)*(1.+X)+H*H)
      Z=(1.-X)/(H*H*UPO)+(1.+X)/(H*H*UNE)
      B(I,1)=1.-(RHO*H*H/2.)*Z
      L1=1
      ZN2=(X-X*X-H*H)/(H*H*UPO)+(X+X*X+H*H)/(H*H*UNE)
      ZN1=LOGF(1.-X+UPO)+(X*X-X*X*X-X*H*H-H*H)/(H*H*UPO)-LOGF(-1.-X+UNE)
     1+(X*X+X*X*X+X*H*H-H*H)/(H*H*UNE)
```

KSU 1410 COMPUTING CENTER

```
      B(I,2)=X*X-(RHO*H*H/2.)*ZN1
      L2=2
      DO 40 J=3,M2
      DO 30 K=1,2
      N=N+1
      R=N
      ZN=1./((R-2.)*UPC)+(2.*R-3.)*X*ZN1/(R-2.)+((R+1.)/(R-2.))*(X*X+H*H
     1)*ZN2-((-1.)**(N-1))/((R-2.)*UNE)-((2.*R-3.)/(R-2.))*X*ZN1-((R+1.)
     2/(R-2.))*(X*X+H*H)*ZN2
      ZN2=ZN1
   30 ZN1=ZN
   40 B(I,J)=X**N-(RHO*H*H/2.)*ZN
   50 X=X+DELX
C     MATRIX SOLUTION FOR IBM 1410  2/64  APPL-RATHFON
  502 FORMAT(2H0 ,E16.8,2I3)
  503 DIMENSION BBB(16,17),ABA(16),BSTAR(16)
  505 FORMAT(2H0 ,2E16.8,I3)
  506 FORMAT(2H0$,E16.8)
  514 FORMAT(2H0 ,5E16.8)
  512 ZK=9.0E10
      LN1=LN+1
      DO 508 LI=1,LN
      ABA(LI)=0.0
  508 BSTAR(LI)=0.0
      ZTRY=0.0
      ZFA=0.0
      ZFAC=1.0
      ZZFAC=1.0
      DO 509 I=1,LN
      ZLL=I
  509 ZZFAC=ZZFAC*ZLL
  510 ZZ=0.0
      DO 547 LI=1,LN
      LIR=LN+1-LI
      IF(B(LIR,LIR))523,521,521
  521 ZY=B(LIR,LIR)-ZAC
      GO TO 524
  523 ZY=-1.*B(LIR,LIR)-ZAC
  524 IF(ZY)525,525,547
  525 IF(LN-LI)526,526,527
  526 LI1=2
      ZZ=1.0
      GO TO 528
  527 LI1=LI+1
  528 CONTINUE
      DO 537 LK=LI1,LN
      IF(ZZ)529,529,530
  529 LK1=LN+1-LK
      GO TO 531
  530 LK1=LK
```

KSU 1410 COMPUTING CENTER

```
  531 IF(B(LK1,LIR))532,533,533
  532 ZY=-1.*B(LK1,LIR)-ZAC
      GO TO 534
  533 ZY=B(LK1,LIR)-ZAC
  534 IF(ZY)535,535,541
  535 IF(ZK-ZY)537,537,536
  536 ZK=ZY
  537 CONTINUE
      IF(ZZ)538,538,540
  538 LI1=LIR+1
      ZZ=1.
      GO TO 528
  539 FORMAT(1H0,23H ALL ELEMENTS IN COLUMN,I3,21H ARE SMALLER THAN ZAC)
  540 ZK=ZK+ZAC
      WRITE OUTPUT TAPE 6,539,LIR
      WRITE OUTPUT TAPE 6,514,ZK,ZAC
      ZAC=ZAC/10.
      GO TO 510
C     READY TO INTERCHANGE TWO ROWS
  541 DO 542 LJ=1,LN1
      ZTE=B(LIR,LJ)
      B(LIR,LJ)=B(LK1,LJ)
  542 B(LK1,LJ)=ZTE
      ZFA=ZFA+1.0
      ZFAC=ZFAC*ZFA
      IF(ZZFAC-ZFAC)544,544,545
  543 FORMAT(1H0,17H ZAC DECREASED TO,E16.8,25H IN LIEU OF REARRANGEMENT
     1)
  544 ZAC=ZAC/10.
      WRITE OUTPUT TAPE 6,543,ZAC
  545 IF(ZZ)547,547,546
  546 GO TO 510
  547 CONTINUE
      IF(SENSE SWITCH 1)548,550
  548 DO 549 LI=1,LN
      DO 549 LJ=1,LN1
  549 WRITE OUTPUT TAPE 6,502,B(LI,LJ),LI,LJ
  550 CONTINUE
      DO 551 LI=1,LN
      DO 551 LJ=1,LN1
  551 BBB(LI,LJ)=B(LI,LJ)
C     MACHINE IS DONE REARRANGING AND READY TO SOLVE THE MATRIX
  511 DO 555 LI=1,LN
      ZELD=B(LI,LI)
      DO 552 LJ=LI,LN1
  552 B(LI,LJ)=B(LI,LJ)/ZELD
      DO 555 LK=1,LN
      IF(LK-LI)553,555,553
  553 ZELS=B(LK,LI)
      DO 554 LJ=1,LN1
```

KSU 1410 COMPUTING CENTER

```
  554 B(LK,LJ)=B(LK,LJ)-ZELS*B(LI,LJ)
  555 CONTINUE
      ZTRY=ZTRY+1.0
C     MATRIX IS SOLVED
      IF(SENSE SWITCH 2)556,558
  556 WRITE OUTPUT TAPE 6,514,ZTRY
      DO 557 LI=1,LN
  557 WRITE OUTPUT TAPE 6,502,B(LI,LN1),LI
  558 DO 559 LI=1,LN
  559 ABA(LI)=B(LI,LN1)+ABA(LI)
      CONTINUE
      DO 560 LI=1,LN
      DO 560 LJ=1,LN
      BSTAR(LI)=ABA(LJ)*BBB(LI,LJ)+BSTAR(LI)
  560 B(LI,LJ)=BBB(LI,LJ)
      DO 561 LI=1,LN
      B(LI,LN1)=BBB(LI,LN1)-BSTAR(LI)
  561 BSTAR(LI)=0.0
      IF(ZTRY-ZCORR)562,565,565
  562 ZZE=0.0
      DO 563 LI=1,LN
  563 ZZE=ZZE+B(LI,LN1)
      IF(ZZE)564,565,564
  564 GO TO 511
  565 WRITE OUTPUT TAPE 6,514,ZTRY,ZCORR
      DO 566 LI=1,LN
  566 WRITE OUTPUT TAPE 6,505,ABA(LI),B(LI,LN1),LI
      X=0.0
      RMAX=-1.0E+06
      RMIN=1.0E+06
      DO 150 L=1,NIN1
      N=2
      UPO=SQRTF((1.-X)*(1.-X)+H*H)
      UNE=SQRTF((1.+X)*(1.+X)+H*H)
      Z=(1.-X)/(H*H*UPO)+(1.+X)/(H*H*UNE)
      RES=ABA(1)*(1.-(RHO*H*H/2.)*Z)
      ZN2=(X-X*X-H*H)/(H*H*UPO)+(X+X*X+H*H)/(H*H*UNE)
      ZN1=LOGF(1.-X+UPO)+(X*X-X*X*X-X*H*H-H*H)/(H*H*UPO)-LOGF(-1.-X+UNE)
     1+(X*X+X*X*X+X*H*H-H*H)/(H*H*UNE)
      RES=RES+ABA(2)*(X*X-(RHO*H*H/2.)*ZN1)
      DO 140 J=3,M2
      DO 130 K=1,2
      N=N+1
      R=N
      ZN=1./((R-2.)*UPO)+(2.*R-3.)*X*ZN1/(R-2.)+((R+1.)/(R-2.))*(X*X+H*H
     1)*ZN2-((-1.)**(N-1))/((R-2.)*UNE)-((2.*R-3.)/(R-2.))*X*ZN1-((R+1.)
     2/(R-2.))*(X*X+H*H)*ZN2
      ZN2=ZN1
  130 ZN1=ZN
  140 RES=RES+ABA(J)*(X**N-(RHO*H*H/2.)*ZN)
```

KSU 1410 COMPUTING CENTER

```
      RES=RES-T**4
      IF(RES-RMAX)102,102,101
  101 RMAX=RES
      XMAX=X
  102 IF(RMIN-RES)104,104,103
  103 RMIN=RES
      XMIN=X
      IF(SENSE SWITCH 3)104,150
  104 WRITE OUTPUT TAPE 6,514,RES,X
  150 X=X+DELX1
  105 FORMAT(1H0,6X,4HRMAX,12X,4HXMAX,12X,4HRMIN,12X,4HXMIN)
      WRITE OUTPUT TAPE 6,105
      WRITE OUTPUT TAPE 6,2,RMAX,XMAX,RMIN,XMIN
      IF(SENSE SWITCH 4)106,60
  106 DO 107 LI=1,LN
  107 WRITE OUTPUT TAPE 6,506,ABA(LI)
   60 CONTINUE
  108 CALL EXIT
      STOP
      END
```

APPENDIX C

Listing of Fortran Program to Solve for Bounds

```
KSU 1410 COMPUTING CENTER

               BOP  BOUND RATH
C     DETERMINATION OF UPPER AND LOWER BOUNDS OF BETA(X)
      DIMENSION ALOW(16),AUP(16)
    1 FORMAT(5E16.8)
    2 FORMAT(1H0,5E16.8)
    3 FORMAT(I3)
   15 READ INPUT TAPE 5,3,M
   16 READ INPUT TAPE 5,1,DELX
   17 READ INPUT TAPE 5,1,XQ
   18 READ INPUT TAPE 5,1,XT
   19 READ INPUT TAPE 5,1,H
   20 READ INPUT TAPE 5,1,RHO
   22 READ INPUT TAPE 5,1,ZAC,ZCORR
   23 READ INPUT TAPE 5,1,T
   24 READ INPUT TAPE 5,3,NIN
   25 READ INPUT TAPE 5,3,NDELTA
      M2=(M/2)+1
      DO 9 NDELTA=1,2
   21 READ INPUT TAPE 5,1,DELTA
    5 FORMAT(1H0,7X,4HDELX,13X,2HXQ,15X,2HXT,14X,1HH,14X,3HRHO)
      WRITE OUTPUT TAPE 6,5
      WRITE OUTPUT TAPE 6,2,DELX,XQ,XT,H,RHO
    6 FORMAT(1H0,7X,5HDELTA,11X,3HZAC,13X,5HZCORR,13X,1HT)
      WRITE OUTPUT TAPE 6,6
      WRITE OUTPUT TAPE 6,2,DELTA,ZAC,ZCORR,T
    7 FORMAT(1H0,7X,1HM,5X,3HNIN,5X,6HNDELTA)
      WRITE OUTPUT TAPE 6,7
    8 FORMAT(1H0,5X,I3,5X,I3,6X,I3)
    9 WRITE OUTPUT TAPE 6,8,M,NIN,NDELTA
      READ INPUT TAPE 5,1,DARX
      READ INPUT TAPE 5,3,MEL
      DO 30 N=1,M2
      READ INPUT TAPE 5,1,ALOW(N)
   30 WRITE OUTPUTTAPE6,2,ALOW(N)
      DO 40 N=1,M2
      READ INPUT TAPE 5,1,AUP(N)
   40 WRITE OUTPUT TAPE 6,2,AUP(N)
   10 FORMAT(1H0,2H X,5X,11HLOWER BOUND,5X,11HUPPER BOUND,5X,10HMEAN VAL
     1UE,5X,17HMAX PERCENT ERROR)
      WRITE OUTPUT TAPE 6,10
      X=0.0
      DO 50 I=1,MEL
      BETAL=0.0
      BETAU=0.0
      BETAL=ALOW(1)+BETAL
      BETAU=AUP(1)+BETAU
      DO 60 N=2,M2
      BETAL=ALOW(N)*X**(2*N-2)+BETAL
   60 BETAU=AUP(N)*X**(2*N-2)+BETAU
      BETAM=(BETAU+BETAL)/2.0
```

KSU 1410 COMPUTING CENTER

```
      TAXER=((BETAU-BETAL)/(2.0*BETAL))*100.0
   11 FORMAT(1H0,F4.2,4E16.8)
      WRITE OUTPUT TAPE 6,11,X,BETAL,BETAU,BETAM,TAXER
   50 X=X+DARX
```

DETERMINATION BY BOUNDING PRINCIPLES OF THE COMBINED RADIATION FLUX ALONG THE SURFACE BETWEEN TWO PARALLEL PLATES OF FINITE EXTENT

by

JOSEPH BOWES RATHFON

B.S., Oklahoma State University, 1962

AN ABSTRACT OF

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A method of bounding principles is presented for solving the integral equation which arises in determining the combined radiation flux between two non-black, finite, parallel plates. Convergent upper and lower bounds are obtained in order to approximate the solution to the problem. The method emphasizes the attainment of an approximate solution of known accuracy, is systematic, and is well adapted to analysis on high speed digital computers.

Agreement with existing solutions has not been achieved and hence, additional study is needed.